Week 1

Chapter 1 + Appendix A

- Signals carry information
- Signals are represented by functions
- Systems transform signals
- Systems are represented by functions, implemented as state machines, differential equations, etc.

Motivation

This course is about signals and systems. Consider the following familiar sounds:

- Dial
- Startup
- Answer

REALITY

MODEL

"Signal"
Information in the form of

- audio
- video
- text

Abstract →

Mathematical functions

"System"
Transforms signals for

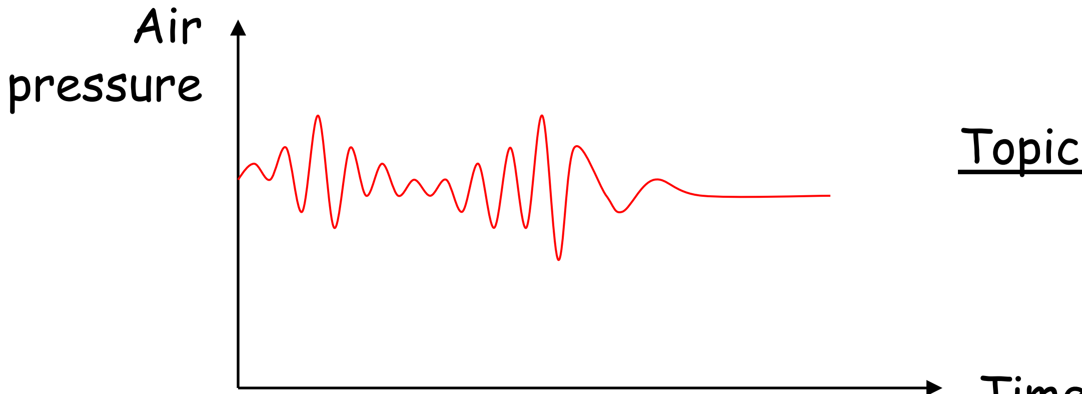
- communication
- computation
- storage
- for control

State machines
Differential equations
Frequency response

← Implement
Predict

Simulate
Calculate
Specify design

Signal



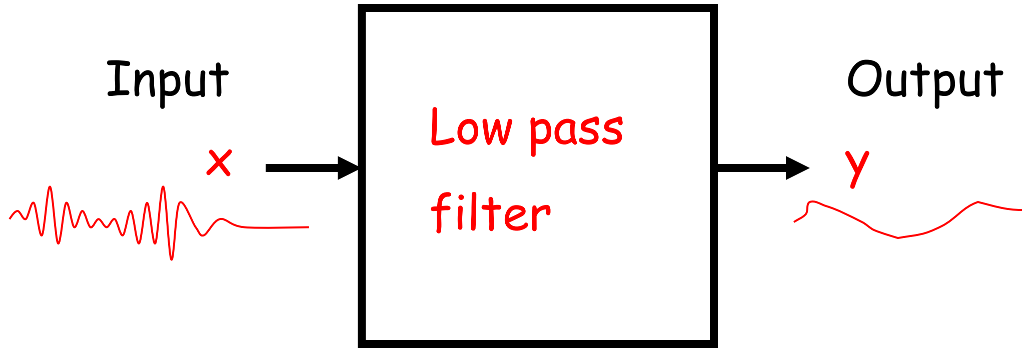
TopicNotes/Signals/Sound

Function description

Sound: Time → Air pressure

TopicNotes/Filtering/Sound

System



System description

Filter: Input signals → Output signals

A signal is a function

A function is specified by four things:

0 the name (f, g, sin, cos, sound, image)

1 the domain (a set)

2 the range (a set)

3 the graph or assignment (for every domain element, a range element)

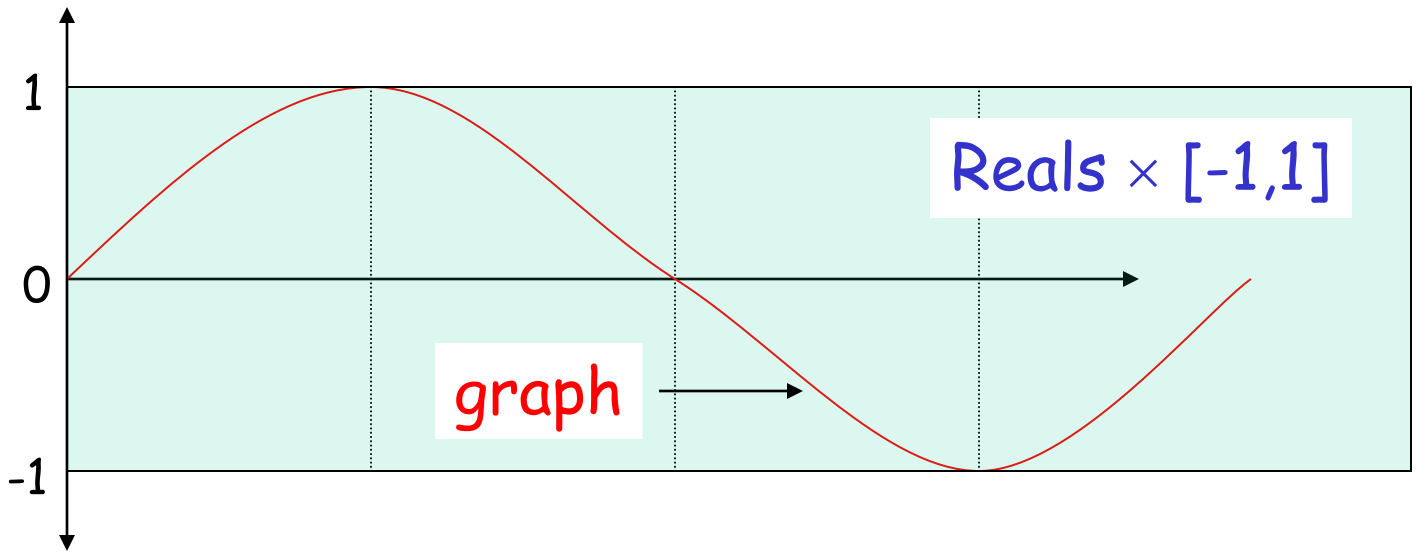
Function examples

sin, cos (names of functions)

1 Domain : Reals = $(-\infty, +\infty)$

2 Range : $[-1,1]$ = $\{ x \in \text{Reals} \mid -1 \le x \le 1 \}$,

3 Graph or Assignment: for each real x, the real $\sin(x) \in [-1,1]$.



Formally, the graph of a function is a set of pairs :

$\{ (x,y) \in \text{Reals} \times [-1,1] \mid y = \sin(x) \}$

$= \{ \ldots , (0,0) , \ldots , (\pi/2, 1) , \ldots , (\pi,0), \ldots , (3\pi/2,-1), \ldots \}$.

$\text{graph} \subset \text{Reals} \times [-1,1]$

Important Mathematical Objects (Appendix A)

1 Sets (unordered collections)

2 Product sets (ordered collections)

3 Functions

Mathematical Language

Let Evens = $\{ x \mid (\exists y,\ y \in Nats \wedge x = 2 \cdot y) \}$.

Mathematical Language

Let Evens = $\{ x \mid (\exists y,\ y \in \text{Nats} \wedge x = 2 \cdot y) \}$.

Constants (names of something)

Mathematical Language

Let Evens = $\{ x \mid (\exists y,\ y \in \text{Nats} \wedge x = 2 \cdot y) \}$.

Constants

Variables (can be replaced by constants)

Mathematical Language

Let Evens = { x | (∃ y, y ∈ Nats ∧ x = 2· y) } .

Constants

Variables

Operators (work on expressions)

Mathematical Language

Let Evens = $\{ x \mid (\exists y,\ y \in Nats \wedge x = 2 \cdot y) \}$.

Constants

Variables

Operators

Quantifiers (of variables)

Mathematical Language

Let Evens = $\{ x \mid (\exists y,\ y \in Nats \wedge x = 2 \cdot y) \}$.

Constants

Variables

Operators

Quantifiers

Definition (LHS = expression)

Defining a new set Evens from known set Nats

Let Evens = { x | (∃ y, y ∈ Nats ∧ x = 2· y) } .

Constants

Variables

Operators

Quantifiers

Definition

Constants have meaning i.e. they name something

20	a certain number
Berkeley	a certain city
false	a certain truth value

Variables have no meaning

but can be substituted by constants

x

y_0

z'

Operators on numbers

number + number	Result: number
number !	number
number = number	truth value
number ≤ number	truth value

Operators on numbers

number + number	Result: number
number !	number
number = number	truth value
number ≤ number	truth value

in matlab

```
>> 3 == 5
ans = 0 (false)
>> 3 == 3
ans =1 (true)
```

assertion

assignment

Operators on cities

merge (city, city)	Result: city
population-of (city)	number
has-a-university (city)	truth value

Operators on truth values {true, false}

truth value $\wedge$ truth value	Result: truth value
truth value $\vee$ truth value	truth value
$\neg$ truth value	truth value
truth value $\Leftrightarrow$ truth value	truth value
truth value $\Rightarrow$ truth value	truth value

Expressions of constants have meaning

	Result
$3 + 20$	Result: 23
$(3! + 2) \cdot 4$	32
$4 \leq$ population-of (Berkeley)	true
$4 \cdot 20 \leq 4 + 20$	false
true $\wedge$ false	false
true $\wedge$ (4 + 20)	not well-formed

Implication

true $\Rightarrow$ true	Result: true
true $\Rightarrow$ false	false
false $\Rightarrow$ true	true
false $\Rightarrow$ false	true

Expressions of variables have no meaning

$x + 20$	Free variables: x
$(3! + y) \cdot 4$	y
$x \leq y$	x, y

Quantifiers remove free variables from expressions

	Result:
$\exists x, \; x = 0$	true
$\forall x, \; x = 0$	false
$\exists y, \; x + 1 = y$	free x
$\forall x, \; \exists y, \; x + 1 = y$	true
$\forall x, \; \exists y, \; x \vee y$	true
$\forall x, \; x + 7$	not well-formed

Every mathematical expression

1. is not well-formed ("type mismatch"), or
2. contains free variables, or
3. is a definition, or
4. has a meaning (e.g., 20, Berkeley, false).

SETS

Set constants

{ 1, 2, 3 }

{ Atlanta, Berkeley, Chicago, Detroit }

{ 1, 2, 3, 4, … }

Famous Sets

Reals	Set of all real numbers
$Reals_+$	Set of all nonnegative real numbers
Ints	Set of all integers
$Ints_+$	Set of all nonnegative integers
Nats	Set of all positive integers {1, 2, ...}
$Nats_0$	Set of all nonnegative integers ($Integers_+$)
Bools	{true, false}
Char	set of all alphanumeric characters
Complex	set of all complex numbers

Set operator

element $\in$ set Result: truth value

$2 \in \{ 1, 2, 3 \}$ true

$2 \in \{ \text{Atlanta}, \text{Berkeley} \}$ false

Set quantifier

($\exists$ x, predicate)	Result: truth value
($\forall$ x, predicate)	truth value
{ x | predicate }	set

Meaning of constants can be defined

Let Nats = { 1, 2, 3, 4, ... } .

Let Bools = { true, false } .

Define Cities = { Atlanta, Chicago, Berkeley, Detroit } .

Define Ø = { } .

explicitly as here,

or implicitly as here

Let Evens = $\{ x \in \text{Nats} \mid \exists\, y \in \text{Nats},\ x = 2 \cdot y \}$.

Let Evens be the set of all $x \in$ Nats such that $x = 2 \cdot y$ for some $y \in$ Nats.

Additional operators on sets

set $\cap$ set	Result: set
set $\cup$ set	set
set \ set	set
set $\subseteq$ set	truth value
set = set	truth value
P (set)	set

Meaning of additional operators can be defined

$\forall$ set X, $\forall$ set Y, let $X \cap Y = \{ z \mid z \in X \wedge z \in Y \}$.

$\forall$ set X, $\forall$ set Y, let $X \cup Y = \{ z \mid z \in X \vee z \in Y \}$.

$\forall$ set X, $\forall$ set Y, let $X \setminus Y = \{ z \mid z \in X \wedge z \notin Y \}$.

$\forall$ set X, $\forall$ set Y, let $X \subseteq Y \Leftrightarrow (\forall z \mid z \in X \Rightarrow z \in Y)$.

$\forall$ set X, $\forall$ set Y, let $X = Y \Leftrightarrow X \subseteq Y \wedge Y \subseteq X$.

$\forall$ set X, let $P(X) = \{ Y \mid Y \subseteq X \}$.

Product sets

Product sets

$set_1 \times set_2$ Result: set of pairs

$set_1 \times set_2 \times set_3$ set of triples

$set_1 \times set_2 = \{ (v,w) \mid v \in set_1 \wedge w \in set_2 \}$

$set_1 \times set_2 \times set_3 = \{ (u,v,w) \mid u \in set_1 \wedge v \in set_2 \wedge w \in set_3 \}$

tuples is a generic name for pairs, triples, …, n-tuples

Product set examples

Set of pixels on an old VGA monitor (640x480), VGAScreen

VGAScreen = {1, 2, ..., 640} x {1, 2, ..., 480}

VGA is Video Graphics Array

The monitor on this laptop is 1400x1050

FUNCTIONS

Each function has four things:

0 the name

1 the domain (a set)

2 the range (a set)

3 the graph (for every domain element,
a range element)

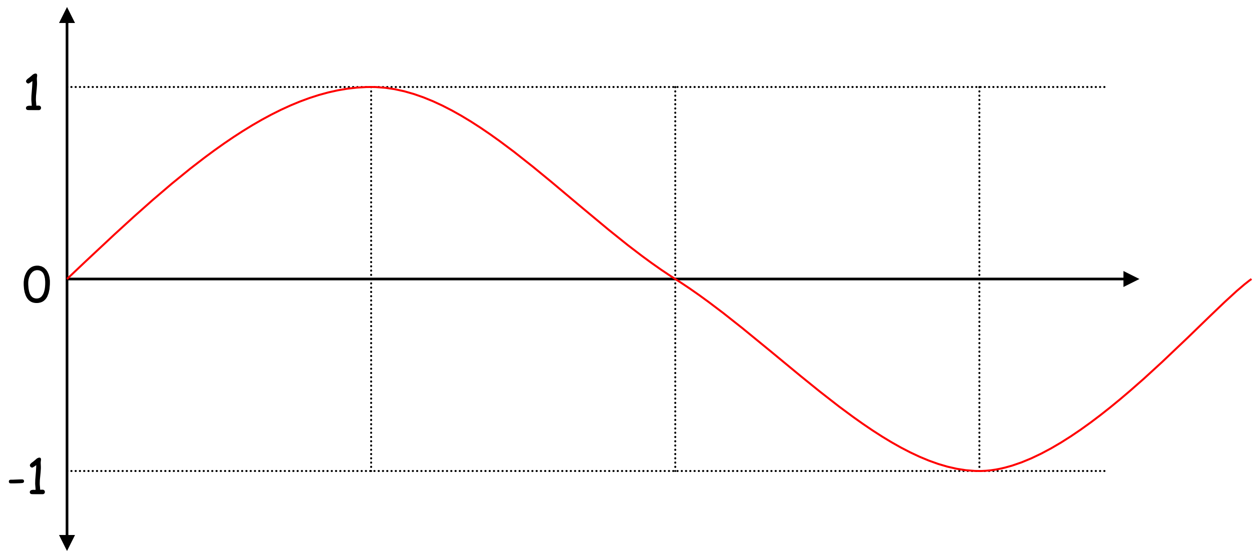
Function examples

sin, cos

1 Domain : Reals .

2 Range : [-1,1] = { x ∈ Reals | -1 ≤ x ≤ 1 } .

3 Graph : for each real x, the real sin (x) ∈ [-1,1] .



Formally, the graph of a function can be thought of as a set of pairs :

$$\{ (x,y) \in (\text{Reals} \times [-1,1]) \mid y = \sin (x) \}$$

$$= \{ \ldots , (0,0) , \ldots , (\pi/2, 1) , \ldots , (\pi,0), \ldots , (3\pi/2,-1), \ldots \} .$$

If domain and range of a function are finite, then the graph can be given by a table :

x	y	$f(x,y) = x \wedge y$
true	true	true
true	false	false
false	true	false
false	false	false

Function definition

Let $f : \text{Domain} \rightarrow \text{Range}$ such that

$\forall\, x \in \text{Domain},\ f(x) = \ldots$

Let twice : Nats $\rightarrow$ Nats such that

$\forall$ x $\in$ Nats, twice (x) = 2$\cdot$ x .

domain (twice) = Nats

range (twice) = Nats

graph (twice) = { (1,2), (2,4), (3,6), (4,8), ... }

Let power : $Nats^2 \rightarrow Nats$ such that

$\forall\, x, y \in Nats$, power$(x, y) = x^y$.

domain (power) = $Nats^2$

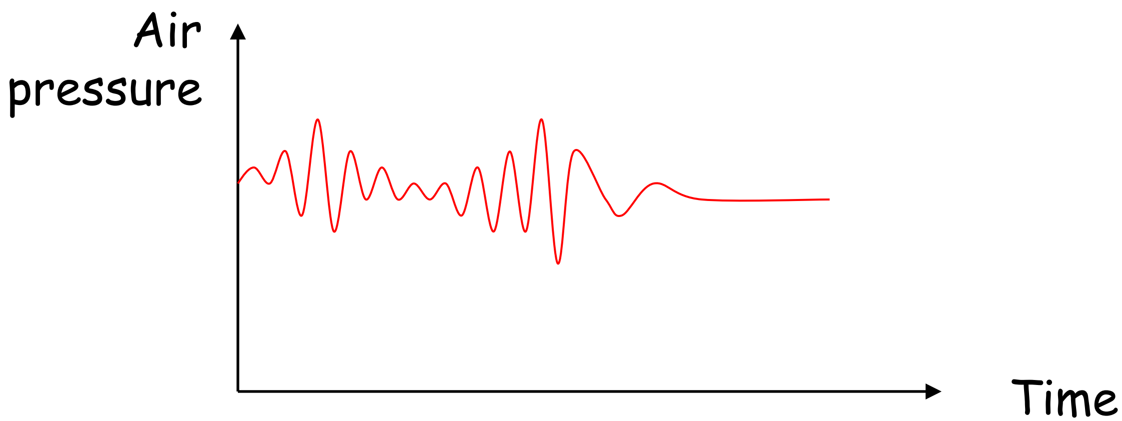
range (power) = Nats

graph (power) = { ((1,1) , 1), ... , ((2,3) , 8), ... }

Let MySound be the signal or function

MySound: Time → Air pressure

such that graph(MySound) is:



Two very important definitions

[set → set]	Result: set of functions
function ∘ function	function

Their meaning

$\forall$ set X, Y,

$[X \to Y] = \{ f \mid \text{domain}(f) = X \wedge \text{range}(f) = Y \}$.

$\forall f \in [X \to Y], \forall g \in [Y \to Z]$,

$g \circ f : X \to Z$ such that

$\forall u \in X, \ (g \circ f)(u) = g(f(u))$.